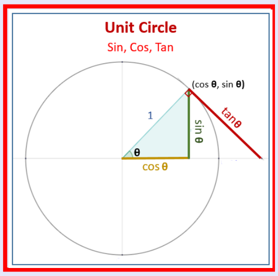
Math 241

Winter 2024

Lecture 16



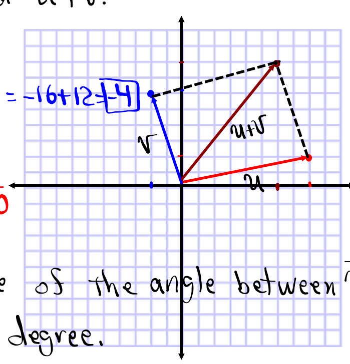
Feb 19-8:47 AM

Given $u = \langle 8, 2 \rangle$, $v = \langle -2, 6 \rangle$

1) $u + v = \langle 8, 2 \rangle + \langle -2, 6 \rangle = \langle 8 + {}^{-}2,\ 2 + 6 \rangle$

$= \langle 6, 8 \rangle$

2) Draw u, v, and $u + v$.



3) $u \cdot v = \langle 8,2 \rangle \cdot \langle -2, 6 \rangle = -16 + 12 = \boxed{-4}$

$|u| = \sqrt{8^2 + 2^2} = \sqrt{68}$

$|v| = \sqrt{(-2)^2 + 6^2} = \sqrt{40}$

4) Find the measure of the angle between $\vec{u}$ & $\vec{v}$

Round to whole degree.

$\cos\theta = \dfrac{u \cdot v}{|u|\,|v|}$ $\quad \cos\theta = \dfrac{-4}{\sqrt{68}\sqrt{40}}$ $\quad \cos\theta = -.077$

$\theta = \cos^{-1}(-.077)$

$\boxed{\theta \approx 94^\circ}$

Jan 30-8:00 AM

Given $u = 6i + 2j$, $v = 3i + 4j$

1) find $u+v = 6i + 2j + 3i + 4j = \boxed{9i + 6j}$

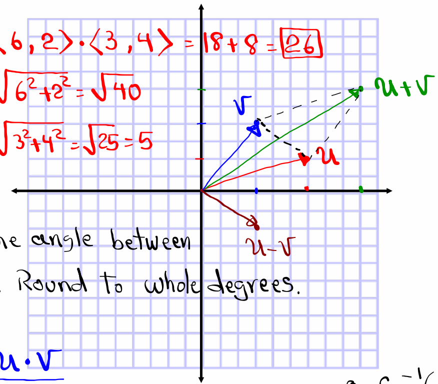
2) find $u - v = 6i + 2j - 3i - 4j = \boxed{3i - 2j}$

3) Draw u, v, $u+v$, and $u - v$.

4) $u \cdot v = \langle 6, 2\rangle \cdot \langle 3, 4\rangle = 18 + 8 = \boxed{26}$

$|u| = \sqrt{6^2 + 2^2} = \sqrt{40}$

$|v| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$



5) Find the angle between $\vec{u}$ & $\vec{v}$. Round to whole degrees.

$$\cos\theta = \frac{u \cdot v}{|u||v|}$$

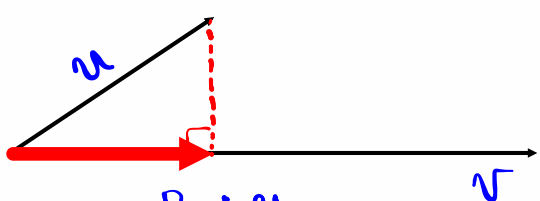
$$\cos\theta = \frac{26}{\sqrt{40} \cdot 5} \qquad \cos\theta \approx .822$$

$\theta = \cos^{-1}(.822)$

$\boxed{\theta \approx 35^\circ}$

Jan 30-8:12 AM

More on Vectors:



$\text{Proj}_v u$ Projection of $\vec{u}$ onto $\vec{v}$

$$\text{Proj}_v u = \left(\frac{u \cdot v}{|v|^2}\right) v$$

$$= \left(\frac{u \cdot v}{|v|}\right)\left(\frac{v}{|v|}\right)$$

Component of u onto v

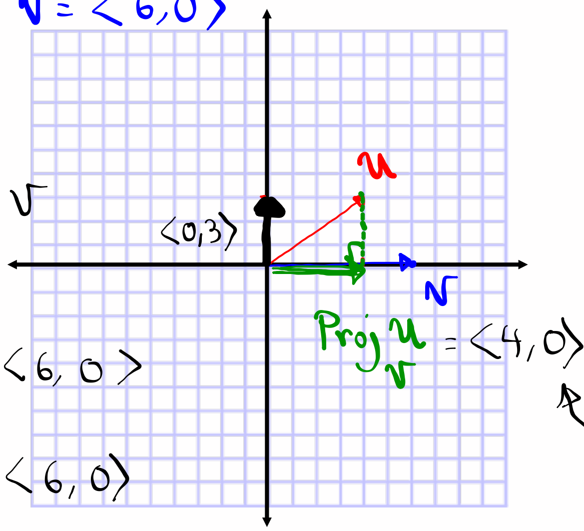
Unit Vector in the direction of v.

Jan 30-8:26 AM

$u = \langle 4, 3 \rangle$, $v = \langle 6, 0 \rangle$

using formula

$$\text{Proj}_v u = \frac{u \cdot v}{|v|^2} v$$



$$= \frac{24}{6^2} \langle 6, 0 \rangle$$

$$= \frac{24}{36} \langle 6, 0 \rangle$$

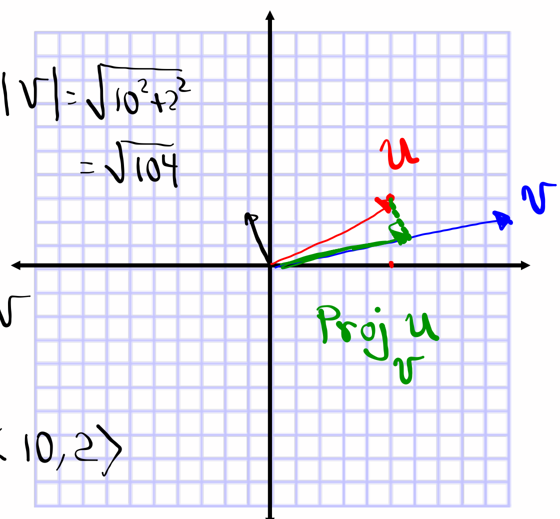
$$= \frac{2}{3} \langle 6, 0 \rangle = \left\langle \frac{2}{3} \cdot 6 , \frac{2}{3} \cdot 0 \right\rangle = \langle 4, 0 \rangle$$

Jan 30-8:32 AM

$u = \langle 5, 3 \rangle$

$v = \langle 10, 2 \rangle$

$$|v| = \sqrt{10^2 + 2^2}$$
$$= \sqrt{104}$$



$$\text{Proj}_v u = \frac{u \cdot v}{|v|^2} v$$

$$= \frac{56}{104} \langle 10, 2 \rangle$$

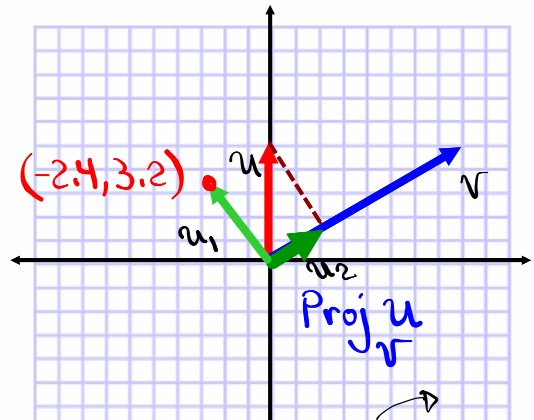
$$= \frac{14}{26} \langle 10, 2 \rangle = \frac{7}{13} \langle 10, 2 \rangle = \left\langle \frac{70}{13} , \frac{14}{13} \right\rangle$$

Jan 30-8:39 AM

$u = \langle 0, 5 \rangle$

$v = \langle 8, 6 \rangle$

Draw u & v



$u_1 + u_2 = u$

$u_1 = u - u_2$

$$\text{Proj}_v u = \frac{u \cdot v}{|v|^2} v$$

$$= \frac{30}{10^2} \langle 8, 6 \rangle$$

$$= \frac{3}{10} \langle 8, 6 \rangle = \left\langle \frac{24}{10}, \frac{18}{10} \right\rangle = \langle 2.4, 1.8 \rangle$$

$$u_1 = \langle 0, 5 \rangle - \langle 2.4, 1.8 \rangle = \boxed{\langle -2.4, 3.2 \rangle}$$

Jan 30-8:46 AM

$u = \langle -2, 9 \rangle$ $\quad v = \langle -1, 2 \rangle$

1) Find component of u along v.

$$\text{Proj}_v u = \frac{u \cdot v}{|v|^2} v = \frac{u \cdot v}{|v|} \frac{v}{|v|}$$

Unit vector in the direction of $\vec{v}$

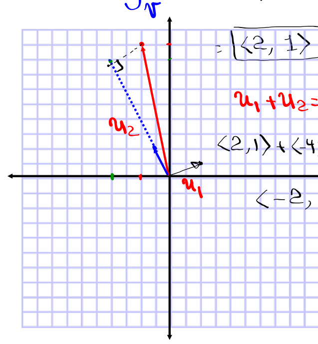
Component of u along $v \rightarrow \frac{u \cdot v}{|v|} = \frac{20}{\sqrt{5}}$

2) Find $\text{Proj}_v u = \frac{u \cdot v}{|v|^2} v = \frac{20}{5} \langle -1, 2 \rangle$

$= \boxed{\langle -4, 8 \rangle}$

3) Find $u - \text{Proj}_v u = \langle -2, 9 \rangle - \langle -4, 8 \rangle$

$= \boxed{\langle 2, 1 \rangle}$



$u_1 + u_2 = u$

$\langle 2, 1 \rangle + \langle -4, 8 \rangle = \langle -2, 9 \rangle$

$\langle -2, 9 \rangle = \langle -2, 9 \rangle$

Jan 30-8:56 AM

$u = \langle -4, 4 \rangle$

$v = \langle 6, 0 \rangle$

Find $\text{Proj}_v u = \dfrac{u \cdot v}{|v|^2} v = \dfrac{-24}{36} \langle 6, 0 \rangle = \boxed{\langle -4, 0 \rangle}$

Find the angle between $\vec{u}$ and $\vec{v}$.

$\cos\theta = \dfrac{u \cdot v}{|u||v|}$

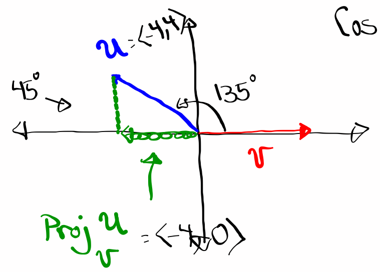
$\cos\theta = \dfrac{-24}{\sqrt{32} \cdot 6} = \dfrac{\cancel{-24}}{\sqrt{16}\sqrt{2} \cdot 6} = \dfrac{-1}{\sqrt{2}} = -\dfrac{\sqrt{2}}{2}$

$\cancel{4}\sqrt{2} \cdot \cancel{6}$

$\cos\theta = -\dfrac{\sqrt{2}}{2}$

R.A. 45° QII $180^\circ - 45^\circ$ QIII $180^\circ + 45^\circ$

$\boxed{135^\circ}$, 225°



SG 21 √

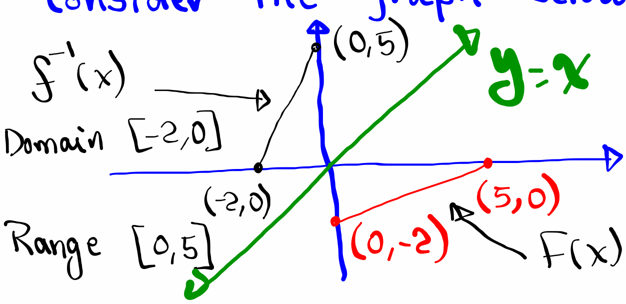
Jan 30-9:09 AM

More with inverse functions

Consider the graph below

Domain $[0, 5]$

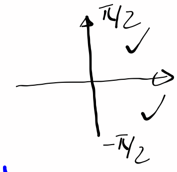
Range $[-2, 0]$

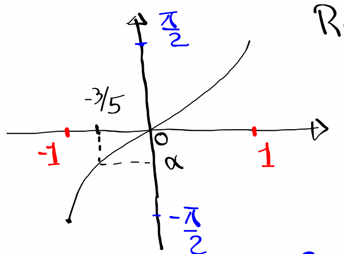


Jan 30-9:39 AM

$y = \sin^{-1} x$ Domain $[-1, 1]$

Range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

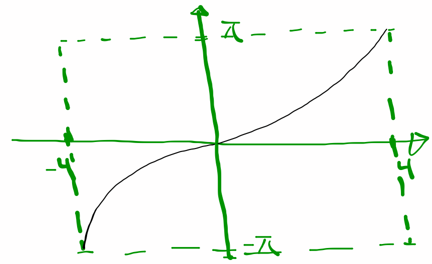




Graph $y = 2\sin^{-1}\left(\frac{1}{4}x\right)$

$-1 \le \frac{1}{4}x \le 1$

$-4 \le x \le 4$

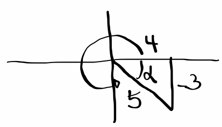


Find exact value of

$\sin\left(2\sin^{-1}\frac{-3}{5}\right)$

$\alpha = \sin^{-1}\frac{-3}{5} \Rightarrow \sin\alpha = \frac{-3}{5}$

$-\frac{\pi}{2} < \alpha < 0 \rightarrow$ Q IV



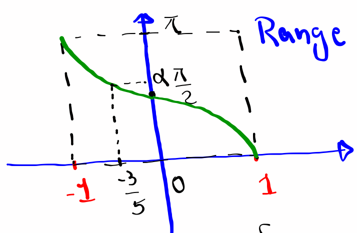
$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2 \cdot \frac{-3}{5} \cdot \frac{4}{5} = \boxed{\frac{-24}{25}}$

Jan 30-9:43 AM

$y = \cos^{-1} x$ Domain $[-1, 1]$

Range $[0, \pi]$



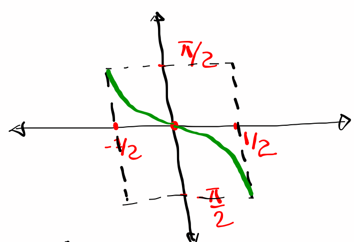
Graph $y = \cos^{-1}(2x) - \frac{\pi}{2}$ shift down

$-1 \le 2x \le 1$

$-\frac{1}{2} \le x \le \frac{1}{2}$

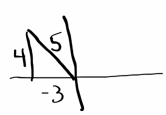
Find exact value for

$\cos\left(\frac{1}{2}\cos^{-1}\frac{-3}{5}\right)$



$\alpha = \cos^{-1}\frac{-3}{5}$

$\cos\alpha = \frac{-3}{5}$



α is in QII → $90° < \alpha < 180°$

$45° < \frac{\alpha}{2} < 90°$

$\cos\left(\frac{1}{2}\alpha\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$

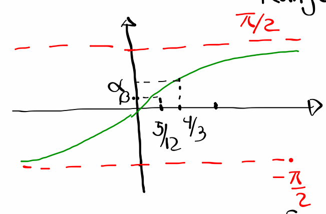
$\frac{\alpha}{2}$ is in QI

$= \sqrt{\frac{1 + {-3/5}}{2}} = \sqrt{\frac{5-3}{10}} = \sqrt{\frac{2}{10}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}} = \boxed{\frac{\sqrt{5}}{5}}$

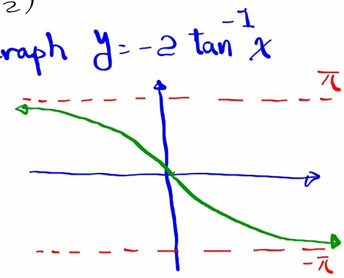
Jan 30-9:53 AM

$y = \tan^{-1} x$ Domain $(-\infty, \infty)$

Range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



Graph $y = -2\tan^{-1} x$



Find exact value of

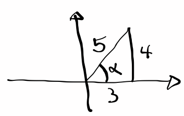
$$\cos\left(\underbrace{\tan^{-1}\frac{4}{3}}_{\alpha} - \underbrace{\tan^{-1}\frac{5}{12}}_{\beta}\right) = \cos(\alpha - \beta)$$

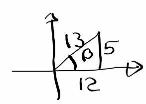
$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\alpha = \tan^{-1}\frac{4}{3}$ $\quad$ $\beta = \tan^{-1}\frac{5}{12}$ $\quad$ $= \frac{3}{5}\cdot\frac{12}{13} + \frac{4}{5}\cdot\frac{5}{13}$

$\tan\alpha = \frac{4}{3}$ $\quad$ $\tan\beta = \frac{5}{12}$

α is in QI $\quad$ β is in QI $\quad$ $= \frac{36}{65} + \frac{20}{65} = \frac{56}{65}$



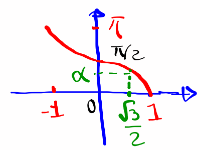


Jan 30-10:04 AM

Find exact value of

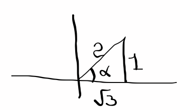
$$\tan\left(\underbrace{\cos^{-1}\frac{\sqrt{3}}{2}}_{\alpha} - \underbrace{\sin^{-1}\frac{-3}{5}}_{\beta}\right)$$

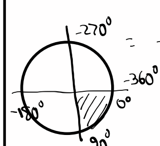
Look at $\cos^{-1} x$



$$= \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

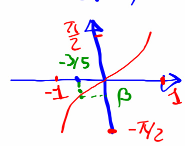
α is in QI





$$= \frac{\frac{\sqrt{3}}{3} - \frac{-3}{4}}{1 + \frac{\sqrt{3}}{3}\cdot\frac{-3}{4}}$$

Now $\sin^{-1} x$



LCD = 12

$$= \frac{4\sqrt{3} + 9}{12 - 3\sqrt{3}} \cdot \frac{12 + 3\sqrt{3}}{12 + 3\sqrt{3}}$$

β is QIV

$$= \frac{48\sqrt{3} + 36 + 108 + 27\sqrt{3}}{144 + 36\sqrt{3} - 36\sqrt{3} - 27} = \frac{144 + 75\sqrt{3}}{117}$$

$$= \frac{\cancel{3}(48 + 25\sqrt{3})}{\cancel{3}\cdot 39} = \boxed{\frac{48 + 25\sqrt{3}}{39}}$$

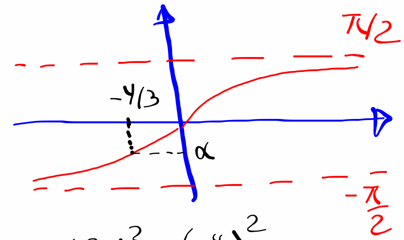
Jan 30-10:18 AM

Find exact value of

$\cos\left(2\tan^{-1}\frac{-4}{3}\right) = \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

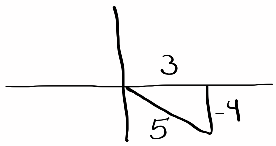
$\alpha = \tan^{-1}\frac{-4}{3}$

$y = \tan^{-1}x$



$\tan\alpha = \frac{-4}{3}$

α is in QIV



$= \left(\frac{3}{5}\right)^2 - \left(\frac{-4}{5}\right)^2$

$= \frac{9}{25} - \frac{16}{25} = \frac{-7}{25}$

$= \boxed{-.28}$

Use your calc to evaluate

$\cos\left(2\tan^{-1}\frac{-4}{3}\right) =$

Jan 30-10:32 AM

Find the exact value of

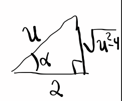
$\csc\left(\underbrace{\csc^{-1}\sqrt{2}}_{\alpha}\right) = \sqrt{2}$

$\alpha = \csc^{-1}\sqrt{2}$ $\rightarrow$ $\csc\alpha = \sqrt{2}$

$\csc\alpha = \sqrt{2}$

Simplify $\sin\left(2\underbrace{\sec^{-1}\frac{u}{2}}_{\alpha}\right)$ $u > 0$

$\alpha = \sec^{-1}\frac{u}{2}$ $\rightarrow$ $\sec\alpha = \frac{u}{2}$ $\rightarrow$ $\cos\alpha = \frac{2}{u}$



$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2\cdot\frac{\sqrt{u^2-4}}{u}\cdot\frac{2}{u} = \boxed{\frac{4\sqrt{u^2-4}}{u^2}}$

In trig. $\rightarrow$ $\sin^{-1}x = \arcsin x$

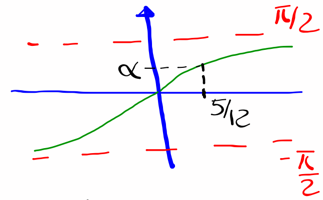
$\cos^{-1}x = \arccos x$

$\tan^{-1}x = \arctan x$

Jan 30-10:39 AM

Find exact value of

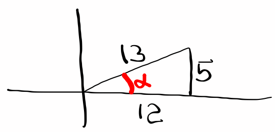
$$\sin\left(\frac{1}{2}\tan^{-1}\frac{5}{12}\right)$$



$\alpha = \tan^{-1}\frac{5}{12}$

$\tan\alpha = \frac{5}{12}$

α is in QI



$0 < \alpha < 90^\circ$

$0 < \frac{1}{2}\alpha < 45^\circ$

$$\sin\left(\frac{1}{2}\alpha\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$= \sqrt{\frac{1-\frac{12}{13}}{2}}$$

LCD = 13

$$= \sqrt{\frac{13-12}{26}}$$

$$= \frac{1}{\sqrt{26}} = \boxed{\frac{\sqrt{26}}{26}}$$

Jan 30-10:49 AM

Complex Numbers:

$a + bi$

a: Real Part

bi: Imaginary Part

$i = \sqrt{-1}$

$i^2 = -1$

ex: $3 - 2i$ Real Part 3
Im. Part -2

ex: $-5 + 4i$ Re. Part -5
Im. Part 4

ex: $-3i$ Re. Part 0, Im. Part -3

Jan 30-11:16 AM

$z_1 = 3 - 2i \ , \quad z_2 = -2 + 5i$

$z_1 + z_2 = 3 - 2i \ - 2 + 5i = \boxed{1 + 3i}$

$z_1 - z_2 = 3 - 2i - (-2 + 5i)$

$= 3 - 2i \ + 2 - 5i = \boxed{5 - 7i}$

$z_1 z_2 = (3 - 2i)(-2 + 5i)$

Foil

$= -6 \ + 15i \ + 4i \ - 10i^2$

$= -6 \ + 19i \ - 10(-1)$

$= -6 \ + 19i \ + 10 = \boxed{4 + 19i}$

Jan 30-11:19 AM

$z_1 = 4 - 3i \qquad z_2 = 4 + 3i$

Find

$z_1 + z_2$

$= 4 - 3i + 4 + 3i = \boxed{8}$

$z_1 - z_2$

$= 4 - 3i - (4 + 3i)$

$= 4 - 3i - 4 - 3i = \boxed{-6i}$

$z_1 \cdot z_2 = (4 - 3i)(4 + 3i)$

$= 16 \ \cancel{+ 12i} \ \cancel{- 12i} \ - 9i^2$

$= 16 - 9(-1) = 16 + 9 = \boxed{25}$

Jan 30-11:24 AM

If $z = a + bi$, then $\bar{z} = a - bi$

z and $\bar{z}$ are conjugate of each other.

$z = -3 - 5i$ $\qquad$ $\bar{z} = -3 + 5i$

Find $z\bar{z} = (-3-5i)(-3+5i)$

$= 9 \cancel{-15i} \cancel{+15i} - 25i^2$

$= 9 - 25(-1) = 9 + 25 = \boxed{34}$

Jan 30-11:27 AM

How to divide complex numbers:

$$\frac{z}{w} = \frac{z \cdot \bar{w}}{w \cdot \bar{w}}$$

Simplify

Final ans in $\boxed{a + bi \text{ Form}}$

$$\frac{5}{1+2i} = \frac{5\,(1-2i)}{(1+2i)(1-2i)} = \frac{5 - 10i}{1 \cancel{-2i} \cancel{+2i} - 4i^2}$$

$$= \frac{5 - 10i}{1 - 4(-1)}$$

$$= \frac{5 - 10i}{5} = \frac{5}{5} - \frac{10}{5}i$$

$$= \boxed{1 - 2i}$$

Jan 30-11:31 AM

Divide $\dfrac{-10i}{3-i}$

$$\frac{-10i}{3-i} = \frac{-10i\,(3+i)}{(3-i)(3+i)} = \frac{-30i - 10i^2}{9 + \cancel{3i} - \cancel{3i} - i^2}$$

$$= \frac{-30i - 10(-1)}{9 - (-1)} = \frac{-30i + 10}{10} = \frac{-30i}{10} + \frac{10}{10}$$

$$= -3i + 1$$

$$= \boxed{1 - 3i}$$

Jan 30-11:35 AM

Divide $\dfrac{2+3i}{2-3i} = \dfrac{(2+3i)(2+3i)}{(2-3i)(2+3i)}$

$$= \frac{4 + 6i + 6i + 9i^2}{4 + \cancel{6i} - \cancel{6i} - 9i^2}$$

$$= \frac{4 + 12i + 9(-1)}{4 - 9(-1)} = \frac{-5 + 12i}{13}$$

$$= \boxed{\frac{-5}{13} + \frac{12}{13}i}$$

Jan 30-11:40 AM

If $z = a + bi$, then $|z| = \sqrt{a^2+b^2}$

Given $z = 6 - 8i$

Find $|z| = \sqrt{6^2+(-8)^2} = \sqrt{100} = \boxed{10}$

Given $z = -12 - 5i$

1) Find $\bar{z} = \boxed{-12 + 5i}$

2) Find $|z| = \sqrt{(-12)^2+(-5)^2} = \sqrt{144+25} = \sqrt{169} = \boxed{13}$

Jan 30-11:44 AM

Powers of i

Simplify $i^{100} = (i^2)^{50} = (-1)^{50} = \boxed{1}$

Simplify $i^{50} = (i^2)^{25} = (-1)^{25} = \boxed{-1}$

Simplify $i^{25} = i^{24} \cdot i$

$= (i^2)^{12} \cdot i = (-1)^{12} \cdot i = 1 \cdot i = \boxed{i}$

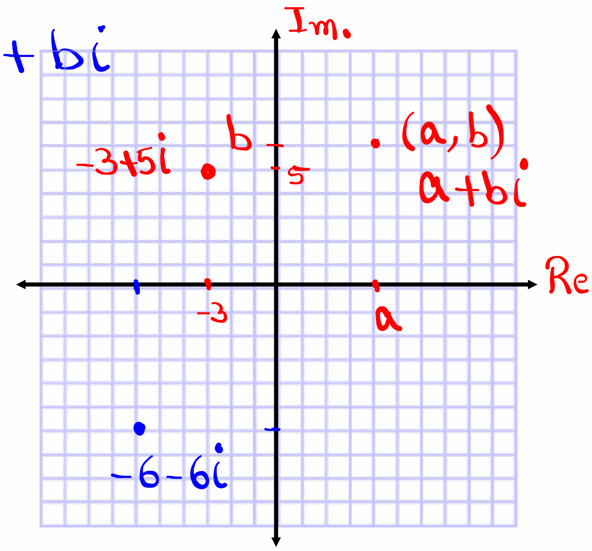
Simplify i^{35}

$= i^{34} \cdot i$

$= (i^2)^{17} \cdot i = (-1)^{17} \cdot i = \boxed{-i}$

Jan 30-11:49 AM

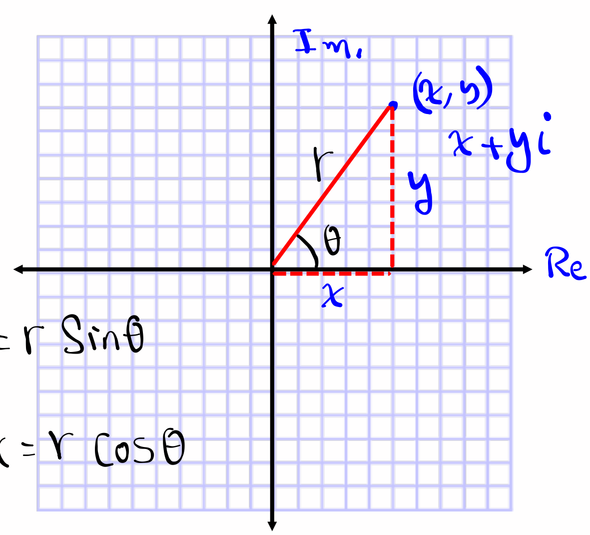
How to plot $a + bi$



Plot $\underset{\text{Re.}}{-3} + \underset{\text{Im.}}{5}i$

Plot $-6-6i$

Jan 30-11:57 AM



$Z = \underset{\text{Re.}}{x} + \underset{\text{Im.}}{y}i$

$\sin\theta = \dfrac{y}{r} \rightarrow y = r\sin\theta$

$\cos\theta = \dfrac{x}{r} \rightarrow x = r\cos\theta$

$$Z = x + yi = \boxed{r\cos\theta + r\sin\theta\, i}$$

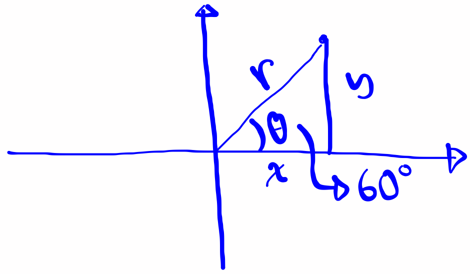
Jan 30-12:01 PM

$Z = \frac{1}{2} + \frac{\sqrt{3}}{2} i$

$r = |Z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \sqrt{\frac{1}{4} + \frac{3}{4}}$

$= \sqrt{\frac{4}{4}} = \sqrt{1} = 1$



$\boxed{r = 1}$

$x = r \cos\theta \qquad \frac{1}{2} = 1 \cdot \cos\theta \qquad \cos\theta = \frac{1}{2} \qquad \theta = 60^\circ$

$y = r \sin\theta \qquad \frac{\sqrt{3}}{2} = 1 \cdot \sin\theta \qquad \sin\theta = \frac{\sqrt{3}}{2} \qquad \theta = 60^\circ$

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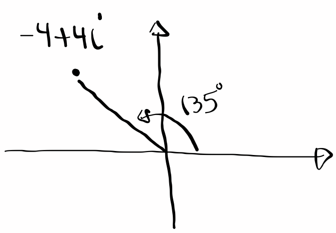
$Z = -4 + 4i$

$r = \sqrt{x^2 + y^2}$

$= \sqrt{(-4)^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$



$x = r\cos\theta$

$-4 = 4\sqrt{2}\cos\theta \Rightarrow \cos\theta = -\frac{\sqrt{2}}{2}$

R.A. 45°

$\theta = 180^\circ - 45^\circ = 135^\circ$

$\frac{-\cancel{4}}{\cancel{4}\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$y = r\sin\theta \qquad 4 = 4\sqrt{2}\sin\theta$

$\sin\theta = \frac{\sqrt{2}}{2}$

R.A. 45° $\qquad \theta = 135^\circ$

$-4 + 4i = r\cos\theta + r\sin\theta\, i$

$= \boxed{4\sqrt{2}\cos 135^\circ + 4\sqrt{2}\sin 135^\circ\, i}$

Make Sure to do/finish SG 21. & Submit.

Jan 30-12:10 PM